b) En déduire que : $\forall n \in \mathbb{N},\ I_{n+1} = \frac{\sqrt{2}}{2n+3} - \left(\frac{2n+2}{2n+3}\right) I_n$.

on

$$I_{n+1} = \sqrt{2} - (2n+2) \cdot \int_0^1 u^{2n+1} \cdot \sqrt{x^2+1}\, dx$$

$$= \sqrt{2} - (2n+2) \cdot \int_0^1 \frac{u^{2n+1} \cdot (u^2+1)}{\sqrt{x^2+1}}\, dx$$

$$= \sqrt{2} - (2n+2) \cdot \int_0^1 \frac{u^{2n+3}}{\sqrt{x^2+1}} + \frac{u^{2n+1}}{\sqrt{x^2+1}}\, dx$$

$$= \sqrt{2} - (2n+2) \cdot \int_0^1 \frac{u^{2n+3}}{\sqrt{x^2+1}}\, dx - (2n+2) \cdot \int_0^1 \frac{u^{2n+1}}{\sqrt{x^2+1}}\, dx$$

$$I_{n+1} = \sqrt{2} - (2n+2) I_{n+1} - (2n+2) I_n$$

$$I_{n+1} + (2n+2) I_{n+1} = \sqrt{2} - (2n+2) I_n$$

$$I_{n+1} (2n+3) = \sqrt{2} - (2n+2) I_n$$

$$\Rightarrow \boxed{I_{n+1} = \frac{\sqrt{2}}{2n+3} - \frac{2n+2}{2n+3} I_n}$$